The Curve Exclusion Theorem for elliptic and K3 fibrations birational to Fano 3-fold hypersurfaces

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Abstract

The theorem referred to in the title is a technical result that is needed for the classification of elliptic and K3 fibrations birational to Fano 3-fold hypersurfaces in weighted projective space. We present a complete proof of the Curve Exclusion Theorem, which first appeared in the author's unpublished PhD thesis [Ry02] and has since been relied upon in solutions to many cases of the fibration classification problem. We give examples of these solutions and discuss them briefly.

1 Introduction

The problem that motivates the work presented here is the following.

1.1 Problem. Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a Fano 3-fold weighted hypersurface in one of the 'famous 95' families of Fletcher and Reid [Fl00]. Assuming that X is general in its family, we seek to classify the set of K3 fibrations $g\colon Z \to T$ with Z birational to X and the set of elliptic fibrations $g\colon Z \to T$ with Z birational to X.

Solutions to both the K3 and elliptic cases of this problem for families 1 and 3 of the 95 first appeared in papers of Cheltsov (see [Ch00], [Ch03] and further references therein). These are the only two of the 95 families whose

members are smooth: $X = X_4 \subset \mathbb{P}^4$ in family 1 is a smooth quartic 3-fold and $X = X_6 \subset \mathbb{P}(1,1,1,1,3)$ in family 3 is a double cover of $\mathbb{P}^3$ branched in a smooth sextic. For four of the 93 remaining singular families, solutions to both the K3 and elliptic cases of Problem 1.1 appeared in [Ry06] and one other case, family 5, was dealt with earlier in the unpublished [Ry02]. Here is an example solution.

1.2 Theorem ([Ry06]). *Let $X = X_{30} \subset \mathbb{P}(1,4,5,6,15)_{x,y,z,t,u}$ be a general member of family 75 of the 95.*

(a) Suppose $\Phi\colon X \dashrightarrow Z/T$ is a birational map from X to a K3 fibration $g\colon Z \to T$ (see 1.11 below for our assumptions on K3 fibrations, and also on elliptic fibrations). Then there exists an isomorphism $\mathbb{P}^1 \to T$ such that the diagram below commutes, where $\pi = (x^4, y)\colon X \dashrightarrow \mathbb{P}^1$.

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "\pi"'] & Z \arrow[d, "g"] \\ \mathbb{P}^1 \arrow[r, "\simeq"] & T \end{tikzcd}$$

(b) There does not exist an elliptic fibration birational to X.

(c) If $\Phi\colon X \dashrightarrow Z$ is a birational map from X to a Fano 3-fold Z with canonical singularities then Φ is actually an isomorphism (so in particular $Z \simeq X$ has terminal singularities).

The proof of this theorem relies on one particular case of our Curves Exclusion Theorem (1.5 below); [Ry06] contains a proof of this case, but no others.

In [Ch05] Cheltsov, building on previous joint work with Park [CP] and on [Ry02], was able to classify elliptic fibrations birational to a general member of any of the 95 families, i.e., to solve completely the elliptic case of Problem 1.1. Both [CP] and [Ch05] rely on Theorem 1.5: see below. One important observation in these two papers — which also appears in a simple form in [Ch00] — is that surprisingly useful information can be extracted from the trivial fact that, in the elliptic case, the linear system on X with which we are working is not a pencil (see, e.g., Lemma 2.1 of [CP] and the proof of Lemma 2.11); largely because of this observation, these papers deal

only with the elliptic case of the classification problem. It should be noted, though, that [CP], building on [Ry02], contains constructions of K3 fibrations birational to general members of all 95 families: it is the problem of excluding other possible K3 fibrations that remains open, for the moment, in most cases.

Here is a theorem from [CP] which relies on our Theorem 1.5.

1.3 Theorem ([CP, 1.2]). *A general $X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ in family N of the 95 is birational to an elliptic fibration if and only if*

$$N \notin \{3,\ 60,\ 75,\ 84,\ 87,\ 93\}.$$

Theorem 1.5 is used in the proof of this result to help demonstrate the nonexistence of a birational elliptic fibration for $N \in \{3,\ 60,\ 75,\ 84,\ 87,\ 93\}$. Similarly, our theorem is used throughout [Ch05] (see his Theorem 1.15 and Lemma 1.16) to classify elliptic fibrations birational to all the 95 families. We give one example:

1.4 Theorem ([Ch05, 26.3]). *Let $X = X_{18} \subset \mathbb{P}(1,1,4,6,7)_{x_0,\,x_1,\,y,\,z,\,t}$ be a general member of family 36 of the 95 and assume that $\Phi\colon X \dashrightarrow Z$ is a birational map from X to an elliptic fibration $g\colon Z \to T$. Then either there exists a birational map $\mathbb{P}(1,1,4) \dashrightarrow T$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "\pi"'] & Z \arrow[d, "g"] \\ \mathbb{P}(1,1,4) \arrow[r, dashed] & T \end{tikzcd}$$

commutes, where $\pi = (x_0, x_1, y)\colon X \dashrightarrow \mathbb{P}(1,1,4)$ is the natural projection, or there exists a birational map $\mathbb{P}(1,1,6) \dashrightarrow T$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "\pi'"'] & Z \arrow[d, "g"] \\ \mathbb{P}(1,1,6) \arrow[r, dashed] & T \end{tikzcd}$$

commutes, where $\pi' = (x_0, x_1, z)\colon X \dashrightarrow \mathbb{P}(1,1,6)$ is the natural projection.

It is time to state the Curve Exclusion Theorem; first we need the following.

Notation. Let X be a normal complex projective variety, $\mathcal{H}$ a mobile linear system on X and $\alpha \in \mathbb{Q}_{\geqslant 0}$. We denote by $\mathrm{CS}(X, \alpha\mathcal{H})$ the set of centres on X of valuations that are strictly canonical or worse for $K_X + \alpha\mathcal{H}$ — that is,

$$\mathrm{CS}(X, \alpha\mathcal{H}) = \{\mathrm{Centre}_X(E) \mid a(E, X, \alpha\mathcal{H}) \leqslant 0\}.$$

This notation is standard. We also use the following nonstandard notation: if $K_X + \alpha\mathcal{H}$ is canonical then $\mathrm{V}_0(X, \alpha\mathcal{H})$ denotes the set of valuations (or of the corresponding divisors, each on some sufficiently blown up model) which are strictly canonical for $K_X + \alpha\mathcal{H}$.

1.5 Curve Exclusion Theorem. *Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x_0,\dots,x_4}$ be a general hypersurface in one of the 95 families and $C \subset X$ a reduced, irreducible curve. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and $C \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then there exists a pair ℓ, ℓ' of linearly independent forms of degree 1 in $(x_0, \dots, x_4)$ such that*

$$C \ \subset \ \{\ell = \ell' = 0\} \cap X. \tag{1}$$

1.6. For a precise discussion of how this theorem is used in the proofs of Theorems 1.2, 1.3 and 1.4 we refer to the papers already cited; but we give a brief outline here. Suppose we have a birational map $\Phi\colon X \dashrightarrow Z/T$ from a Fano 3-fold hypersurface $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ in one of the 95 families to either an elliptic or a K3 fibration $g\colon Z \to T$. By an analogue of the Noether–Fano–Iskovskikh inequalities — which are used in the Sarkisov program to break up a birational map between two Mori fibre spaces into elementary links (see [Co95]) — the log pair $\left(X, \frac{1}{n}\mathcal{H}\right)$ has nonterminal singularities, where $\mathcal{H} = \Phi_*^{-1} g^*|A_T|$ is the transform on X of a very ample complete linear system $|A_T|$ on T and $n = \deg \mathcal{H}$ is its anticanonical degree, i.e., $\mathcal{H} \subset |-nK_X|$. Using the main theorem of [CPR], which states that $X = X_d$ is *birationally rigid*, we reduce to the case where $\left(X, \frac{1}{n}\mathcal{H}\right)$ has canonical but nonterminal, i.e., strictly canonical, singularities.

At this point it is natural to ask what $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ is; one of the main results we use to answer this is our Curve Exclusion Theorem 1.5, which tells us that the only curves that could be in $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ are the obvious ones (see below). Of course we also need results describing which nonsingular and singular points could belong to $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$, but we do not discuss

this here. Finally, given a complete list of possibilities for $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$, we use various techniques to try to deduce a complete list of birational elliptic and K3 fibrations (this is a simplification of the process, but it gives the general idea).

We expand a little on why it is 'obvious' that certain curves cannot be excluded. We need the following:

1.7 Proposition ([Ry02, 2.2]). *Let $X_d \subset \mathbb{P}(1,1,a_2,a_3,a_4)$ be general in one of the families with $a_1 = 1$ and $\ell, \ell' \in k[x_0,\dots,x_4]$ two independent forms of degree 1. Then a general fibre S of $\pi = (\ell,\ell')\colon X \dashrightarrow \mathbb{P}^1$ is a quasismooth Du Val K3 surface and, setting $\mathcal{P} = \pi_*^{-1}\left|\mathcal{O}_{\mathbb{P}^1}(1)\right|$,*

$$\mathrm{CS}(X,\mathcal{P}) = \{C_0,\dots,C_r,P_1,\dots,P_s\},$$

where $C_0,\dots,C_r$ are the components of $\{\ell = \ell' = 0\} \cap X$ and $P_1,\dots,P_s$ are all the singularities of X.

We do not prove this result fully here, but make two remarks. Firstly, it is clear in principle that, in the above statement, a general $S \in \mathcal{P}$ is a quasismooth Du Val K3 surface (though in fact in the case $a_2 = 1$ it is not immediate that S is quasismooth: we are allowed a general X and a general $S \in \mathcal{P}$ but must prove the result for every possible $\mathcal{P}$, not just a general choice). Secondly, it is clear that $C_0,\dots,C_r \in \mathrm{CS}(X,\mathcal{P})$. This shows that Theorem 1.5 excludes as many curves as it possibly could.

We say no more about how 1.5 is used to solve cases of Problem 1.1: see [Ry06], [Ch05] and [Ry02] for details.

Contents of this paper

The remaining sections of the present paper are devoted to proving Theorem 1.5. The proof requires several different methods and explicit checks of dozens of cases, so often there is no choice but to give an example calculation and a list of other cases that are similar, together with case-specific choices that need to be made. We have therefore thought it best to split the material up into sections according to the type of exclusion argument used. The first of these, Section 2, contains arguments that are coarse and elementary — really they are just lemmas about curves of low degree in weighted projective

4-space — but they still dispose of a large number of families. Sections 3 and 4 then deal with the curves that slipped through the net, of which there are many more than one might wish. The arguments of Section 4 are generally more fiddly than those of Section 3, and they are also required for a good many more cases; these are summarised in Table 1.

Conventions and assumptions

Our notations and terminology are mostly as in, for example, [KM], but we list here some conventions that are nonstandard, together with assumptions that will hold throughout.

1.8. All varieties considered are complex, and they are projective and normal unless otherwise stated.

1.9. All curves are reduced and irreducible unless otherwise stated.

1.10 The famous 95 families. These are ordered as in [Fl00] and [CPR], and we assume known the basic facts about them such as quasismoothness, $\mathrm{Cl}\, X \simeq \mathbb{Z}$, etc. We choose coordinates (x, y, z, t, u) or (x, y_1, y_2, z, t) etc. in order of ascending degree, again as in [CPR] — for example, in the case of family 36, as we saw in Theorem 1.4, we choose (x_0, x_1, y, z, t) as coordinates for $\mathbb{P}(1, 1, 4, 6, 7)$. If v is a coordinate then P_v denotes the point where only v is nonzero. We import from [CPR] the notion of a *starred monomial assumption* — for example, if $X = X_{15} \subset \mathbb{P}(1, 1, 3, 4, 7)_{x_0, x_1, y, z, t}$ is a member of family 25, we make the assumption $*tz^2$, i.e., we assume that tz^2 appears with nonzero coefficient in the defining equation of X. Whenever X is a member of one of the 95 families we let $A = -K_X = \mathcal{O}_X(1)$ denote the positive generator of the class group; moreover, if $f\colon Y \to X$ is a birational morphism then B denotes $-K_Y$.

1.11 Definitions. Let Z be a normal projective variety with canonical singularities. A *fibration* is a morphism $g\colon Z \to T$ to another normal projective variety T such that $\dim T < \dim Z$ and $g_*\mathcal{O}_Z = \mathcal{O}_T$. We say g is an *elliptic fibration*, resp. a *K3 fibration*, if and only if its general fibre is an elliptic curve, resp. a K3 surface.

1.12. Usually when we write an equation explicitly or semi-explicitly in terms of coordinates we omit scalar coefficients of monomials; this is the 'coefficient convention'.

1.13. If the letter n is used without explicit definition, it refers to the degree of the mobile linear system $\mathcal{H}$ on X, as in the statement of Theorem 1.5.

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2 Coarse numerics and curves of low degree

Our first lemma uses the standard argument to bound the degree of a curve centre.

2.1 Lemma. *Let X be any hypersurface in one of the 95 families and $C \subset X$ a curve, reduced but possibly reducible. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and each irreducible component of C belongs to* $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. *Then* $\deg C = AC \leqslant A^3$.

PROOF. Let s be a natural number such that sA is Cartier and very ample, and pick general members $H, H' \in \mathcal{H}$. Now by assumption

$$\mathrm{mult}_{C_i}(H) = \mathrm{mult}_{C_i}(H') = n$$

for each irreducible component C_i of C, so for a general $S \in |sA|$

$$A^3 sn^2 = SHH' \geqslant sn^2 AC = sn^2 \deg C,$$

which proves that $\deg C \leqslant A^3$. □

It is now necessary to understand the geometry of curves of low degree, i.e., degree at most A^3, lying inside our $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$. The statement of Theorem 1.5 suggests the following natural case division.

CASE 1: $a_1 > 1$. $|\mathcal{O}_X(1)| = \langle x_0 \rangle$ is fixed so there do not exist two independent degree 1 forms ℓ, ℓ'; therefore we are trying to exclude *all* curves. Lemma 2.2 below shows that for many families with $a_1 > 1$ there are in fact no curves of degree at most A^3 inside X, other than (perhaps) curves contracted by $\pi_4 \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$; so for these families we have already nearly won. There are five families with $a_1 > 1$ to which Lemma 2.2 does not apply, and we also need to consider curves contracted by π_4 — see Lemma 2.5, which applies also to many families with $a_1 = 1$, although there are exceptional cases both with $a_1 > 1$ and with $a_1 = 1$ that fail to satisfy the hypotheses.

CASE 2: $a_1 = 1$ AND $a_2 > 1$. $|\mathcal{O}_X(1)| = \langle x_0, x_1 \rangle$ is a pencil so we are trying to exclude all curves not contained in $\{x_0 = x_1 = 0\} \cap X$. Lemma 2.3 below shows that for many of these families any curve $C \subset X$ that is not contracted by π_4 and not contained in $\{x_0 = x_1 = 0\} \cap X$ has degree larger than A^3, so it is excluded by Lemma 2.1. Again there are families that have $a_1 = 1$ and $a_2 > 1$ but fail to satisfy the hypothesis — in fact there are twelve such families — and, as already mentioned, curves contracted by π_4 are considered separately in Lemma 2.5.

CASE 3: $a_0 = a_1 = a_2 = 1$. Families with $\dim |\mathcal{O}_X(1)| \geqslant 2$ are dealt with in the next section.

2.2 Lemma. *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a hypersurface in one of the families with $a_1 > 1$ and suppose that either*

(a) $d < a_1 a_4$ or

(b) $d < a_2 a_4$ and the curve $\{x = y = 0\} \cap X$ is irreducible (which holds for general X in a family with $a_1 > 1$ by Bertini's theorem).

Then any curve $C \subset X$ that is not contracted by $\pi_4 \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ has $\deg C > A^3$. Consequently C is excluded absolutely by Lemma 2.1.

Remark. Out of the families with $a_1 > 1$, numbers 18, 19, 22, 27 and 28 have $d \geqslant a_2 a_4$, so that, as written here, this lemma fails to deal with them. (In fact we shall see in Section 4 that the conclusion of the lemma is true for them as well.) Of the remainder, many have $a_1 a_4 \leqslant d < a_2 a_4$, which means that part (b) of the lemma applies to them under the generality assumption

stated; this happens for numbers 23, 32, 33, 37, 38, 39, 42, 43, 44, 48, 49, 52, 55, 56, 59, 63, 64, 65, 72, 73, 77 and 89. For the rest, the stronger form (a) applies and no extra generality assumption is needed: numbers 40, 45, 57, 58, 60, 61, 66, 68, 69, 74, 75, 76, 78, ..., 81, 83, ..., 87 and 90, ..., 95.

Proof of 2.2. Most of the following proof has already appeared in [Ry06], but we reproduce it here for the convenience of the reader. The part that is not in [Ry06] is the discussion of the cases where assumption (2) below fails to hold.

So suppose, contrary to the statement of the lemma, that $C \subset X$ has $\deg C \leqslant A^3$ and is not contracted by π_4; let $C' \subset \mathbb{P}(1, a_1, a_2, a_3)$ be the set-theoretic image $\pi_4(C)$. Note that $\deg C' \leqslant \deg C$ — indeed, if H denotes the hyperplane section of $\mathbb{P}(1, a_1, \ldots, a_4)$ and H' that of $\mathbb{P}(1, a_1, a_2, a_3)$, we pick $s \geqslant 1$ such that $|sH|$ and $|sH'|$ are very ample, and calculate that

$$\begin{aligned} s \deg C \;&=\; (sH)C \;=\; \pi_4^*(sH')C \\ &=\; sH'(\pi_4)_*C \;=\; srH'C' \;=\; sr \deg C' \;\geqslant\; s \deg C', \end{aligned}$$

where $r \geqslant 1$ is the degree of the induced morphism $\pi_4|_C \colon C \to C'$. So in fact $\deg C$ is a multiple of $\deg C'$ by the positive integer r. (The point of $|sH|$ being very ample is that we can move it away from P_4, where π_4 is undefined, and apply the projection formula to the morphism $\pi_4|_{\mathbb{P}(1,a_1,\ldots,a_4) \smallsetminus \{P_4\}}$.)

Now we form the diagram below.

$$\begin{array}{ccc} C & \subset & \mathbb{P}(1, a_1, a_2, a_3, a_4) \\ \downarrow & & \downarrow{\scriptstyle \pi_4} \\ C' & \subset & \mathbb{P}(1, a_1, a_2, a_3) \\ \downarrow & & \downarrow{\scriptstyle \pi_3} \\ \{*\} & \subset & \mathbb{P}(1, a_1, a_2) \end{array}$$

C' is contracted by π_3 — indeed, if its image were a curve C'' we would have

$$\deg C'' \;\leqslant\; \deg C' \;\leqslant\; \deg C \;\leqslant\; A^3,$$

but $A^3 = d/(a_1a_2a_3a_4) < 1/(a_1a_2)$, since $d < a_3a_4$ in either case (a) or (b), and on the other hand $1/(a_1a_2) \leqslant \deg C''$ simply because $C'' \subset \mathbb{P}(1, a_1, a_2)$ — contradiction.

For convenience we make the following assumption:

$$\text{assume that } (a_1, a_2) = 1. \tag{2}$$

(We discuss at the end of the proof what to do if $(a_1, a_2) > 1$.) (2) implies that the point $\{*\} \subset \mathbb{P}(1, a_1, a_2)$ is, up to coordinate change, one of

$$\{y = z = 0\}, \quad \{y^{a_2} + z^{a_1} = x = 0\}, \quad \{x = z = 0\} \quad \text{and} \quad \{x = y = 0\},$$

using the coefficient convention in $y^{a_2} + z^{a_1} = 0$. It follows that the curve $C' \subset \mathbb{P}(1, a_1, a_2, a_3)$ is defined by the same equations. In the first case, this means that $\deg C' = 1/a_3 > d/(a_1a_2a_3a_4) = A^3$, contradiction. In the second case $\deg C' = 1/a_3$ again, because

$$C' \simeq \{y^{a_2} + z^{a_1} = 0\} \subset \mathbb{P}(a_1, a_2, a_3)$$

passes only through the singularity $(0, 0, 1)$, using (2) — so we obtain a contradiction as in the first case. In the case $C' = \{x = z = 0\}$, we have $\deg C' = 1/(a_1a_3)$ and we easily obtain a contradiction from $a_2a_4 > d$. In the final case, $C' = \{x = y = 0\}$, if the assumptions in part (a) of the statement hold then we have

$$\deg C' = 1/(a_2a_3) > d/(a_1a_2a_3a_4) = A^3,$$

contradiction; while if the assumptions in part (b) hold then

$$C = \{x = y = 0\} \cap X$$

(because the right hand side is irreducible), but

$$\deg\big(\{x = y = 0\} \cap X\big) = a_1A^3 > A^3,$$

since we also assumed $a_1 > 1$ — contradiction.

This completes the proof subject to the assumption (2); we now discuss what to do if it does not hold. First we note that there are only 9 families with $(a_1, a_2) > 1$, namely numbers 18, 22, 28, 43, 52, 59, 69, 73 and 81. The first three of these fail to satisfy either (a) or (b), so we need not concern ourselves with them — though we remark that the argument we are about to give works for number 18 and fails for 22 and 28, with the inequality becoming an

equality. Now consider as an example family 43, $X_{20} \subset \mathbb{P}(1,2,4,5,9)_{x,y,z,t,u}$ with $A^3 = 1/18$, and assume that $\{*\} = \{y^2 + z = x = 0\} \subset \mathbb{P}(1,2,4)$, which is obviously the only problem case. Then

$$C' = \big(\{y^2 + z = x = 0\} \subset \mathbb{P}(1,2,4,5)\big) \simeq \big(\{y^2 + z = 0\} \subset \mathbb{P}(2,4,5)\big),$$

which of course has

$$\deg C' = 1/(a_3 \operatorname{hcf}(a_1,a_2)) = 1/(5 \times 2) = 1/10 > 1/18,$$

contradiction. Exactly the same observation works for numbers 52, 59, 69, 73 and 81: one needs only to check that $1/(a_3 \operatorname{hcf}(a_1,a_2)) > A^3$, which is true in each case. $\square$

2.3 Lemma. *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1,1,a_2,a_3,a_4)$ be a hypersurface in one of the families with $a_1 = 1$ and $a_2 > 1$; suppose that $d < a_2a_4$. Then any curve $C \subset X$ that is not contracted by π_4 and that satisfies $\deg C \leqslant A^3$ is contained in $\{x_0 = x_1 = 0\} \cap X$.*

Remark. Out of the families with $a_1 = 1$ and $a_2 > 1$ this lemma fails to deal with numbers 7, 9, 11, 12, 13, 15, 16, 17, 21, 24, 29 and 34. These require extra work: see Section 4 and particularly Table 1.

Proof of 2.3. Take such a curve C and suppose $C \not\subset \{x_0 = x_1 = 0\}$.

$$\begin{array}{ccc} C & \subset & \mathbb{P}(1,1,a_2,a_3,a_4) \\ \downarrow & & \Big\downarrow{\scriptstyle \pi_4} \\ C' & \subset & \mathbb{P}(1,1,a_2,a_3) \\ \downarrow & & \Big\downarrow{\scriptstyle \pi_3} \\ \{*\} & \subset & \mathbb{P}(1,1,a_2) \end{array}$$

As in Lemma 2.2 the image of C' under π_3 is a point — indeed, if the image were a curve C'' we would have

$$\deg C'' \leqslant A^3 = d/(a_2a_3a_4) < 1/a_2 \leqslant \deg C'',$$

because $d < a_2a_4 \leqslant a_3a_4$ — contradiction. Therefore after coordinate change $C' = \{x_1 = x_2 = 0\}$, since by assumption $C' \neq \{x_0 = x_1 = 0\}$, and so

$$\deg C' = 1/a_3 > d/(a_2a_3a_4) = A^3,$$

contradiction. $\square$

2.4. Now we need to deal with curves contracted by π_4. As discussed in [CPR, 5.6], we can write the equation of X in one of the forms

(a) $x_4^3 + ax_4 + b = 0$, or

(b) $x_4^2 + b = 0$, or

(c) $x_j x_4^2 + ax_4 + b = 0$ (with $j = 1$, 2 or 3),

where $a(x_0, \dots, x_3)$ and $b(x_0, \dots, x_3)$ are weighted homogeneous polynomials of the appropriate degrees. In cases (a) and (b), $\pi_4 \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ is a morphism with finite fibres; in case (c), π_4 contracts a finite set of curves whose union is $\{x_j = a = b = 0\} \subset X$.

2.5 Lemma. *Suppose $X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$ is a general hypersurface in one of the 95 families and assume $d < a_1 a_2 a_3$. Then any curve $C \subset X$ contracted by π_4 has $\deg C > A^3$, and is therefore excluded absolutely by Lemma 2.1.*

Remark. This lemma fails to deal with families such that $P_4 \in X$ and $d \geqslant a_1 a_2 a_3$. These are number 18, which has $a_1 > 1$; numbers 7, 12, 13, 16, 20, 24, 25, 26 and 46, which have $a_1 = 1$ and $a_2 > 1$; and numbers 2, 5 and 8, which have $a_0 = a_1 = a_2 = 1$.

Proof of 2.5. If there exists a contracted curve C then a fortiori the equation of X takes the form (c) of 2.4 above. Consider the subscheme Z of the space $\mathbb{P}^2(a_0, \dots, \widehat{a_j}, \dots, a_3) =: \mathbb{P}(a'_0, a'_1, a'_2)$ defined by $Z = \{a = b = 0\}$, substituting $x_j = 0$ into a and b. Z is a finite set of points (because $a, b \in k[x_{a'_0}, x_{a'_1}, x_{a'_2}]$ have no common factor — see [CPR, 4.5]) and the union of the contracted curves is the cone over Z obtained by varying x_4, still with $x_j = 0$. Below we show that

$$\text{for general } X,\ Z \text{ misses any singular points of } \mathbb{P}(a'_0, a'_1, a'_2); \tag{3}$$

therefore our contracted curve C passes through only one singular point of X, namely P_4. Consequently

$$\deg C \geqslant 1/a_4 > d/(a_1 a_2 a_3 a_4) = A^3$$

as required.

It remains to show (3). We assume $j = 1$ to simplify the notation — no generality is lost in doing so because the proof below does not make use of $a_1 \leqslant a_2 \leqslant a_3$. We know

$$Z = \{a_{d-a_4} = b_d = 0\} \subset \mathbb{P}(1, a_2, a_3)$$

and we need to show that either $a_2 \,|\, (d - a_4)$ or $a_2 \,|\, d$. This demonstrates that $(0, 1, 0) \notin Z$, assuming X is general. Formally we also need to show that either $a_3 \mid (d - a_4)$ or $a_3 \mid d$, but the proof is identical. Note that even if $(a_2, a_3) \neq 1$ the only two points of $\mathbb{P}(1, a_2, a_3)$ which can be singular are $(0, 1, 0)$ and $(0, 0, 1)$.

Now to the proof. Because $x_1x_4^2$ is the tangent monomial to X at P_4, we know that

$$\begin{aligned} a_1 + 2a_4 &= d \qquad \text{and} && (4)\\ a_2 + a_3 &= a_4, && (5) \end{aligned}$$

where (5) follows from (4) and $d = a_1 + \cdots + a_4$. Now we consider the different possibilities for the tangent monomial to X at P_2.

If $x_4x_2^n$ is the tangent monomial to X at P_2 then $a_4 + na_2 = d$, so $a_2 \,|\, (d - a_4)$ and we are done. If $x_3x_2^n$ is the tangent monomial at P_2 then $a_3 + na_2 = d$, so

$$(n - 1)\, a_2 = d - a_4$$

using (5), which shows that $n \geqslant 2$ and $a_2 \,|\, (d - a_4)$ as required. If x_2^n is the tangent monomial then $P_2 \notin X$ and $a_2 \,|\, d$.

We are left with the case $x_1x_2^n$. We know that

$$\begin{aligned} a_1 + na_2 &= d \qquad\qquad \text{and} && (6)\\ a_3 + a_4 &= (n - 1)\, a_2, && (7) \end{aligned}$$

where as before (7) follows from (6) and $d = a_1 + \cdots + a_4$. Now (5) and (7) imply

$$\begin{aligned} 2a_3 &= (n - 2)\, a_2 \qquad \text{and}\\ 2a_4 &= na_2. \end{aligned}$$

If n is even then $a_2 \,|\, a_3$ and $a_2 \mid a_4$, so $a_2 = 1$ (any three of (a_1, a_2, a_3, a_4) have highest common factor 1 because the K3 section $\{x_0 = 0\} \cap X$ is well

formed). Therefore $a_2 \mid d$, as required. If on the other hand n is odd then $a_2 = 2a_2'$ is even and a_2' divides a_2, a_3 and a_4, so $a_2' = 1$ and

$$(a_0, \ldots, a_4) \;=\; (1, a_1, 2, a_4 - 2, a_4)$$

with $a_4 = n$ odd. If a_1 is even then, by (6), d is even and $a_2 = 2 \mid d$; but if a_1 is odd then d is odd as well and $a_2 \;=\; 2 \mid (d - a_4)$. $\square$

3 The test class method

The following lemma is completely general and elementary; we will use it for curves inside X, but it is also important for excluding singular points: see [Ry06, Theorem 3.20]. It should be compared with [CPR, 5.2.1], to which it is closely related.

3.1 Lemma. *Let X be a Fano 3-fold hypersurface in one of the 95 families and $\mathcal{H}$ a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical; suppose $\Gamma \subset X$ is an irreducible curve or a closed point satisfying $\Gamma \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$, and furthermore that there is a Mori extremal divisorial contraction*

$$f\colon (E \subset Y) \to (\Gamma \subset X), \quad \mathrm{Centre}_X\, E = \Gamma,$$

such that $E \in \mathrm{V}_0\left(X, \frac{1}{n}\mathcal{H}\right)$ (for the notation $\mathrm{V}_0\left(X, \frac{1}{n}\mathcal{H}\right)$, see the Introduction). Then $B^2 \in \overline{\mathrm{NE}}\, Y$.

Proof. We know that

$$K_Y + \tfrac{1}{n}\mathcal{H}_Y \;\sim_{\mathbb{Q}}\; f^*\left(K_X + \tfrac{1}{n}\mathcal{H}\right) \;\sim_{\mathbb{Q}}\; 0.$$

It follows that $B \sim_{\mathbb{Q}} \frac{1}{n}\mathcal{H}_Y$, and therefore $B^2 \in \overline{\mathrm{NE}}\, Y$, because $\mathcal{H}_Y$ is mobile. $\square$

The idea of the test class method is very simple. Suppose $\Gamma \subset X$ is an irreducible curve or a closed point that is the centre of a Mori extremal divisorial contraction $f\colon (E \subset Y) \to (\Gamma \subset X)$ as in the above lemma. A *test class* is, by definition, a nonzero nef class $M \in \mathrm{N}^1 Y$.

3.2 Lemma (cf. [CPR, 5.2.3]). *Suppose that, in the situation just described, there is a test class M on Y with $MB^2 < 0$. Then E cannot be a strictly canonical singularity for any $\mathcal{H}$.*

Proof. This is immediate from Lemma 3.1. $\square$

3.3 Corollary. *If the hypotheses of Lemma 3.2 are satisfied by some curve $C = \Gamma \subset X$ then C is excluded absolutely, that is, C is not a strictly canonical centre for any $\mathcal{H}$.*

Proof. We are assuming that there exists a Mori extremal divisorial contraction $f\colon (E \subset Y) \to (C \subset X)$ with $\mathrm{Centre}_X(E) = C$. Suppose $\mathcal{H}$ is mobile of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ strictly canonical. Clearly what we need to prove is the following: if $C \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ then in fact $E \in \mathrm{V}_0\left(X, \frac{1}{n}\mathcal{H}\right)$. To see this, first note that over a general point $P \in C \subset X$, $f\colon Y \to X$ must be the blowup of $\mathcal{I}_C$. Let $P \in S \subset X$ be a general surface through P, smooth near P and transverse to C. Then

$$\mathrm{mult}_P\left(\mathcal{H}|_S\right) = n$$

because $C \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ by assumption and we have the classical fact that, locally over $P = C \cap S \subset S$, the first ordinary blowup extracts a divisor of maximal multiplicity for $\mathcal{H}|_S$. $\square$

The problem with the test class method is that it only applies to curves $C \subset X$ that are centres of Mori extremal divisorial contractions. Such curves are always contained in $\mathrm{Nonsing}(X)$ and their own singularities are also restricted. It turns out that the test class method, together with coarse arguments like those of Section 2, is sufficient to prove Theorem 1.5 for families with $a_0 = a_1 = a_2 = 1$; for the other families, the curves that the coarse results fail to deal with hit singularities of X, and we need other methods.

We now turn to the more practical question of how to find a test class for a given curve.

3.4 Definition (cf. [CPR, 5.2.4]). Let L be a Weil divisor class in a 3-fold X and $\Gamma \subset X$ an irreducible curve or a closed point. We say that L *isolates* Γ, or is a Γ-*isolating class*, if and only if there exists $s \in \mathbb{Z}_{\geqslant 1}$ such that the linear system $\mathcal{L}^s_\Gamma := |\mathcal{I}^s_\Gamma(sL)|$ satisfies

- $\Gamma \in \mathrm{Bs}\,\mathcal{L}^s_\Gamma$ is an isolated component (i.e., in some neighbourhood of Γ the subscheme $\mathrm{Bs}\,\mathcal{L}^s_\Gamma$ is supported on Γ); and

- if Γ is a curve, the generic point of Γ appears with multiplicity 1 in $\operatorname{Bs}\mathcal{L}_\Gamma^s$.

3.5 Lemma. *Suppose that L isolates $\Gamma \subset X$ and let $s \in \mathbb{Z}_{\geqslant 1}$ be as above. Then for any extremal divisorial contraction*

$$f\colon (E \subset Y) \to (\Gamma \subset X) \quad \textit{with} \quad \operatorname{Centre}_X(E) = \Gamma$$

the birational transform $M = f_*^{-1}\mathcal{L}_\Gamma^s$ is a test class on Y.

Proof. This is [CPR, 5.2.5]. □

We now use the test class method, together with some elementary arguments in the style of Lemmas 2.2 and 2.3, to prove Theorem 1.5 for all the families with $a_0 = a_1 = a_2 = 1$, that is, for families $1, \dots, 6, 8, 10$ and 14.

3.6 Proof of Theorem 1.5 assuming $\mathbf{a_0 = a_1 = a_2 = 1}$. Let

$$X \;=\; X_d \;\subset\; \mathbb{P}(1,1,1,a_3,a_4)$$

be a hypersurface in one of families $1, \dots, 6, 8, 10$ and 14 and $C \subset X$ a curve; suppose that C is a strictly canonical centre for some $\mathcal{H}$. By Lemma 2.1, $\deg C \leqslant A^3$.

Case 1: C is contracted by $\pi_4\colon X \dashrightarrow \mathbb{P}(1,1,1,a_3)$. By Lemma 2.5, we are in a family with $d \geqslant a_1a_2a_3$ and $P_4 \in X$, that is, one of families 2, 5 and 8. It is very easy to check in each of these cases that the contracted curves are contained in $\{\ell = \ell' = 0\} \cap X$ for two linearly independent forms ℓ, ℓ' of degree 1 in $(x_0, \dots, x_4)$ — for example, in the case of family 8, $X_9 \subset \mathbb{P}(1,1,1,3,4)_{x_0,x_1,x_2,y,z}$ with $A^3 = 3/4$, we do a coordinate change so that the tangent monomial at $P_4 = P_z$ is x_2z^2; then the equation of X is

$$x_2z^2 + a_5z + b_9 \;=\; 0 \quad \text{with} \quad a, b \in k[x_0, x_1, x_2, y]$$

and the contracted curves are the irreducible components of

$$\{x_2 = a_5 = b_9 = 0\} \;\subset\; \mathbb{P}(1,1,1,3).$$

But $y^3 \in b_9$ by quasismoothness at P_y and therefore after a coordinate change

$$C \;=\; \{x_1 = x_2 = y = 0\} \;\subset\; \{x_1 = x_2 = 0\} \cap X.$$

Case 2: C is not contracted by π_4. As in the proofs of Lemmas 2.2 and 2.3 we consider the following diagram.

$$\begin{tikzcd} C \arrow[d] & \subset & \mathbb{P}(1,1,1,a_3,a_4) \arrow[d, dashed, "\pi_4"] \\ C' \arrow[d] & \subset & \mathbb{P}(1,1,1,a_3) \arrow[d, dashed, "\pi_3"] \\ C'' & \subset & \mathbb{P}(1,1,1) \end{tikzcd}$$

We may assume that C' is not contracted by π_3 — indeed, any point in $\mathbb{P}^2$ is defined by two linearly independent forms ℓ, ℓ' of degree 1 in (x_0, x_1, x_2). So $\deg C'' \in \mathbb{Z}_{\geqslant 1}$ and therefore $\deg C', \deg C'' \in \mathbb{Z}_{\geqslant 1}$ also (they are positive integral multiples of $\deg C''$ — see the proof of Lemma 2.2). For families 8, 10 and 14, $A^3 < 1$ and we already have our contradiction; families $1, \ldots, 6$ remain.

The next step is to show that if C is not contained in some $\{\ell = \ell' = 0\}$ then, after a coordinate change, it is one of the following; here N denotes the number of the family.

1. $N = 1$, $X_4 \subset \mathbb{P}^4$, $A^3 = 4$, $C =$ a twisted cubic curve in some linearly embedded $\mathbb{P}^3 \subset \mathbb{P}^4$, test class $2A - E$.

2. $N = 2$, $X_5 \subset \mathbb{P}(1,1,1,1,2)$, $A^3 = 5/2$, $C = \{y = x_3 = x_0x_1 + x_2^2 = 0\}$, $\deg C = 2$, test class $2A - E$.

3. $N = 3$, $X_6 \subset \mathbb{P}(1,1,1,1,3)$, $A^3 = 2$, $C = \{y = x_3 = x_0x_1 + x_2^2 = 0\}$, $\deg C = 2$, test class $6A - E$.

4. $N = 4$, $X_6 \subset \mathbb{P}(1,1,1,2,2)$, $A^3 = 3/2$, $C = \{y_2 = y_1 = x_0 = 0\}$, $\deg C = 1$, test class $2A - E$.

5. $N = 5$, $X_7 \subset \mathbb{P}(1,1,1,2,3)$, $A^3 = 7/6$, $C = \{z = y = x_0 = 0\}$, $\deg C = 1$, test class $6A - E$.

6. $N = 6$, $X_8 \subset \mathbb{P}(1,1,1,2,4)$, $A^3 = 1$, $C = \{z = y = x_0 = 0\}$, $\deg C = 1$, test class $4A - E$.

As an illustration of how to derive this list we consider family 4 (the others being easier). We have $X = X_6 \subset \mathbb{P}(1,1,1,2,2)_{x_0,x_1,x_2,y_1,y_2}$ with $A^3 = 3/2$; if necessary we do a coordinate change so that $P_4 = P_{y_2} \notin X$. Take a curve $C \subset X$ of degree at most $A^3 = 3/2$ in $\mathbb{P}^2$, so it is a line $\{x_0 = 0\}$ after coordinate change and

$$\deg C = \deg C' = \deg C'' = 1.$$

Now $C' \subset \big(\{x_0 = 0\} \cap \mathbb{P}(1,1,1,2)\big) \simeq \mathbb{P}(1,1,2)$ is an irreducible curve so, after coordinate change, it is $\{y_1 = x_0 = 0\}$. But

$$\{y_1 = x_0 = 0\} \cap X \ \simeq\ \{y_2^3 + a_2 y_2^2 + b_4 y_2 + c_6 \ =\ 0\} \ \subset\ \mathbb{P}(1,1,2)$$

with $a, b, c \in k[x_1, x_2]$. Because C is a degree 1 component of this, it corresponds to a linear factor of the cubic in y_2, so after another coordinate change $C = \{y_2 = y_1 = x_0 = 0\}$, as required.

Finally the curves in the above list need to be excluded using the test class method. The method is essentially the same in each case, so we give the details only for case 3. So let $X = X_6 \subset \mathbb{P}(1,1,1,1,3)_{x_0,\dots,x_3,y}$ be a general member of family 3 and suppose $C = \{y = x_3 = x_0x_1 + x_2^2 = 0\} \subset X$. It is clear that $6A$ is C-isolating (Definition 3.4, using $s = 1$), so by Lemma 3.5 $M = 6A - E$ is a test class, where $f\colon (E \subset Y) \to (C \subset X)$ is the blowup of C. But

$$\begin{aligned} MB^2 &= (6A - E)(A - E)(A - E) \\ &= 6A^3 \ -\ 13A^2E \ +\ 8AE^2 \ -\ E^3 \\ &= 6 \times 2 \ -\ 0 \ -\ 8 \times 2 \ -\ 0 \ =\ -4 \ <\ 0 \end{aligned}$$

so C is excluded by Corollary 3.3. In the calculation we used

$$A^2E \ =\ 0, \quad AE^2 \ =\ -\deg C \ =\ -2,$$

$$E^3 \ =\ -\deg \mathcal{N}_{C|X} \ =\ -\deg C \ +\ 2 \ -\ 2p_a(C) \ =\ 0.$$

This completes the proof. $\square$

4 Surface methods for the remaining curves

The task that remains is to prove Theorem 1.5 for families with $a_2 > 1$. This involves checking lots of cases; before listing them we consider two families in full detail so as to illustrate the two main methods we need.

4.1 Example (Theorem 1.5 for family 20). Take a general

$$X_{13} \subset \mathbb{P}(1,1,3,4,5)_{x_0,x_1,y,z,t} \quad \text{with } A^3 = 13/60;$$

we make two starred monomial assumptions: $*tz^2$ and $*zy^3$. The presence of yt^2, on the other hand, is guaranteed by quasismoothness at P_t, so after a coordinate change we can write the defining equation of X as

$$yt^2 + a_8 t + b_{13} = 0 \quad \text{with} \quad a,\, b \in k[x_0, x_1, y, z].$$

Let C be one of the curves of degree $1/5 < 13/60 = A^3$ which pass through P_t and are flopped by the quadratic involution i_{P_t}: so C is a component of $\{y = a_8 = b_{13} = 0\} \subset X$. The components of this set are the only curves that remain to be excluded for this X — indeed, $d < a_2 a_4$ so Lemma 2.3 applies to curves not contracted by π_4 — and of course all the components of $\{y = a_8 = b_{13} = 0\}$ are the same up to coordinate change, so it is enough to exclude one of them, our C.

After a coordinate change we may assume $C = \{x_0 = y = z = 0\} \subset X$, that is, C is the $x_1 t$-stratum (note that $z^2 \in a_8$ by $*tz^2$, so $P_z \notin C$ before the change). To exclude C we follow the general method described in [CM, §5], taking a general surface $T \in |4A - C|$ and doing the following calculations.

Claim. *(a)* $\operatorname{Bs}|4A - C|$ *is supported on* $C \cup \{P_y\}$.

(b) T *has a* $\frac{1}{5}(1,1)$ *singularity at* $P_t \in C \subset T$ *and* T *is smooth at all other points of* C.

(c) The selfintersection $(C)^2_T = -9/5$.

For the proof, see below. Suppose now that $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and $C \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Restricting to T, we have $\mathcal{H}|_T = nC + \mathcal{L}$ where $\mathcal{L}$ is the mobile part. It

follows that $\left(\frac{1}{n}\mathcal{H}|_T - C\right) \sim_{\mathbb{Q}} \frac{1}{n}\mathcal{L}$ is nef on T; but we calculate

$$\begin{aligned}\left(\tfrac{1}{n}\mathcal{L}\right)_T^2 \;=\; \left(\tfrac{1}{n}\mathcal{H}|_T - C\right)_T^2 \;&=\; (A|_T)^2 \;-\; 2\,(A|_T)\,C \;+\; (C)_T^2 \\ &=\; A^2T \;-\; 2AC \;+\; (C)_T^2 \\ &=\; 4 \times \tfrac{13}{60} \;-\; 2 \times \tfrac{1}{5} \;-\; \tfrac{9}{5} \;=\; -\tfrac{4}{3} \;<\; 0,\end{aligned}$$

contradiction.

PROOF OF CLAIM. (a) A general element $T \in |4A - C|$ has equation

$$z \;+\; yS^1(x_0, x_1) \;+\; x_0\, S^3(x_0, x_1) \;=\; 0, \tag{8}$$

with the coefficient convention. If $P \in \operatorname{Bs}|4A - C|$ then clearly $z = x_0 = 0$ at P; if $y \neq 0$ then $x_1 = 0$ so $a_8 = b_{13} = 0$ because neither contains a pure power of y, and it follows from the defining equation of X that $t = 0$.

(b) Inside X, $P_t \sim \frac{1}{5}(1,1,4)$ in local coordinates (x_0, x_1, z). The usual manipulation of the defining equation of X, together with a local analytic coordinate change, shows that $y = z^2 + x_0^8 + \cdots + x_0\, x_1^7$ near P_t (note that x_1^8 does not appear because $C \subset X$). Therefore a general $T \in |4A - C|$, which is globally defined by (8), is locally defined by

$$z \;+\; \left(z^2 + x_0\, S^7(x_0, x_1)\right) S^1(x_0, x_1) \;+\; x_0\, S^3(x_0, x_1) \;=\; 0,$$

so $(P_t \in T) \sim \frac{1}{5}(1,1)$ in local coordinates (x_0, x_1). Note that near $P_t \in T$ the curve C is defined by $x_0 = 0$.

To show that T is smooth at all other points of C one considers the affine piece $\{x_1 \neq 0\} \subset \mathbb{P}(1,1,3,4,5)$, inside which T is defined by

$$yt^2 + a_8 t + b_{13} \;=\; 0 \quad \text{and} \quad z + y + yx_0 + x_0^4 + \cdots + x_0 \;=\; 0$$

with $a, b \in k[x_0, y, z]$. Writing down the four partial derivatives of each of these two expressions, and evaluating them along $\{x_0 = y = z = 0\}$, one sees that if X is general the rank of the 4×2 matrix never drops below 2.

(c) The nontrivial part here is to calculate the *different*, $\operatorname{Diff} \subset C$, which is the divisor satisfying

$$(K_T + C)\,|_C \;=\; K_C + \operatorname{Diff}.$$

C is Cartier away from $P_t \in T$ so Diff is supported on P_t and the only problem is to calculate the coefficient. We use Corti's result [FA, 16.6.3], which implies that Diff $= \frac{m-1}{m} P_t$ where m is the index of C at $P_t \in T$, provided that $K_T + C$ is plt (purely log terminal) at P_t. But the plt condition is clear in this case: $P_t \in T$ is resolved by the $\frac{1}{5}(1,1)$ (i.e., ordinary) blowup, the discrepancy of K_T is $1/5 - 4/5 = -3/5$ (because $a_E(K_X) = 1/5$ for the $\frac{1}{5}(1,1,4)$ blowup of $P_t \in X$, and T has local weight $4/5$), and $C \subset T$ has local weight $1/5$; so the log discrepancy of $K_T + C$ is $-3/5 - 1/5 = -4/5 > -1$. Clearly $m = 5$, so Diff $= \frac{4}{5} P_t$.

The rest is easy. $T \subset X$ is Cartier in codimension 2, because X has isolated singularities, so

$$\begin{aligned} -2 + \tfrac{4}{5} &= (K_T + C)\, C = (C)^2_T + (K_X + T)\, C \\ &= (C)^2_T + 3AC = (C)^2_T + \tfrac{3}{5} \end{aligned}$$

and therefore $(C)^2_T = -9/5$ as required. □

4.2 Example. Family 29, $X_{16} \subset \mathbb{P}(1,1,2,5,8)_{x_0,x_1,y,z,t}$ with $A^3 = 1/5$. Suppose that X contains the curve $C = \{x_0 = y = t = 0\}$. (An easy argument in the style of the proofs of Lemmas 2.2 and 2.3 shows that up to coordinate change this C is the only curve of degree at most A^3 not contained in $\{x_0 = x_1 = 0\}$.) We can write the equation of X as

$$t^2 + a_8 t + b_{16} = 0 \quad \text{with} \quad a,\, b \in k[x_0, x_1, y, z].$$

We have assumed that $C \subset X$, which means that after making the substitution $x_0 = y = 0$ in a and b we are left with a reducible quadratic $t(t + c_8) = 0$, where $c \in k[x_1, z]$. In other words,

$$\operatorname{Bs} |2A - C| = \{x_0 = y = 0\} \cap X = C + C',$$

where $C' = \{x_0 = y = t + c_8 = 0\}$ is just like C after a coordinate change. Now let $T \in |2A - C|$ be a general surface.

Claim. *(a) T has a $\frac{1}{5}(1,3)$ singularity at P_z and is smooth elsewhere.*

(b) The selfintersection $(C)^2_T = -7/5$ and so, by symmetry, $(C')^2_T = -7/5$ as well.

See below for the proof. Suppose now that $\mathcal{H}$ is mobile of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ canonical and $C \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then, restricting to T, we have $\frac{1}{n}\mathcal{H}|_T \sim_{\mathbb{Q}} C + \alpha C' + \frac{1}{n}\mathcal{L}$, where $0 \leqslant \alpha \leqslant 1$ and $\mathcal{L}$ is the mobile part of $\mathcal{H}|_T$. But $\frac{1}{n}\mathcal{H}|_T \sim_{\mathbb{Q}} A|_T = C + C'$, so $\frac{1}{n}\mathcal{L} \sim_{\mathbb{Q}} (1-\alpha)C'$. It follows that

$$0 \leqslant (\tfrac{1}{n}\mathcal{L})^2_T = (1-\alpha)^2(C')^2_T = -\tfrac{7}{5}(1-\alpha)^2,$$

and therefore $\alpha = 1$.

Consequently $C' \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ as well — but $\deg(C + C') = 2A^3$, contradicting Lemma 2.1.

Proof of Claim. (a) Near to P_z, after a local analytic coordinate change, $T = \{y = 0\}$, so clearly $P_z \sim \frac{1}{5}(1,3)$ inside T. Showing T is smooth elsewhere can be done as in the proof of the claim in Example 4.1 above.

(b) This is also essentially the same as the calculation in Example 4.1. We check that $K_T + C$ is plt at P_z, and it is clear that the index of C at P_z is 5, so by Corti's result [FA, 16.6.3] we have

$$\begin{aligned} -2 + \tfrac{4}{5} = \deg(K_C + \mathrm{Diff}) &= (K_T + C)C = (K_X + T)C + (C)^2_T \\ &= AC + (C)^2_T = \tfrac{1}{5} + (C)^2_T. \end{aligned}$$

The desired conclusion follows. □

4.3 Proof of Theorem 1.5 assuming $\mathbf{a_1} > 1$. For the majority of the families with $a_1 > 1$, Lemmas 2.1, 2.2 and 2.5 prove Theorem 1.5. We need consider only families 18, 19, 22, 27 and 28, which fail to satisfy the hypotheses for Lemma 2.2 — and family 18 also fails to satisfy the hypotheses for Lemma 2.5. The way things turn out is as follows: firstly, for families 19, 22, 27 and 28 there are in fact no curves of degree at most A^3 contained in X, and for family 18 the only curves of degree at most A^3 are those contracted by π_4 — in other words, Lemma 2.2 in fact applies to *all* the families with $a_1 > 1$, provided we make generality assumptions. Secondly, the curves in family 18 contracted by π_4 can be excluded as in Example 4.1, using a general surface $T \in |4A - C|$.

We make no further remarks about the exclusion of curves contracted by π_4 in the case of family 18, but give an example of how to extend Lemma 2.2 to families 18, 19, 22, 27 and 28. So consider family 19,

$$X_{12} \subset \mathbb{P}(1,2,3,3,4)_{x,y,z_1,z_2,t} \text{ with } A^3 = 1/6.$$

Let $P_1, P_2, P_3, P_4 \sim \frac{1}{3}(1,2,1)$ be the singularities on the z_1z_2-stratum and $Q_1, Q_2, Q_3 \sim \frac{1}{2}(1,1,1)$ those on the yt-stratum. We assume that the curve $\{x = y = 0\} \cap X$ is irreducible and that $P_iQ_j \not\subset X$ for all i, j; a general X satisfies these assumptions. Now suppose that C is a curve of degree at most A^3 contained in X. Again we form the following familiar diagram.

$$\begin{tikzcd}
C \arrow[d] & \subset & \mathbb{P}(1,2,3,3,4) \arrow[d, dashed, "\pi_4"] \\
C' \arrow[d] & \subset & \mathbb{P}(1,2,3,3) \arrow[d, dashed, "\pi_3"] \\
C'' & \subset & \mathbb{P}(1,2,3)
\end{tikzcd}$$

Certainly C' is a curve, because $P_t \notin X$. Suppose that C'' is also a curve. Then its degree is $1/6 = A^3$ and it is defined by $\{x = 0\}$ after a coordinate change. Therefore $\deg C = \deg C' = 1/6$ also, and C' is isomorphic to a curve in $\mathbb{P}(2,3,3)$, so after a coordinate change we have $C' = \{x = z_1 = 0\}$. The same argument applied to C now shows that $C = \{x = z_1 = t = 0\}$, after another coordinate change, so $C = P_iQ_j$, contradiction.

Therefore in fact $C'' = \{*\}$ is a point. After a coordinate change, this point is one of

$$\{y = z_1 = 0\}, \quad \{x = y^3 + z_1^2 = 0\}, \quad \{x = z_1 = 0\} \quad \text{and} \quad \{x = y = 0\}.$$

In the first two cases $\deg C' = 1/3 > A^3$, contradiction. In the last case, $C \subset \{x = y = 0\} \cap X$, but the right-hand side has degree $1/3 > A^3$ and is irreducible by assumption — contradiction again. In the third case, an easy argument shows that $C = P_iQ_j$ for some i, j, which gives a contradiction as above.

Similar arguments can be used to extend Lemma 2.2 to families 18, 22, 27 and 28. This completes the proof of Theorem 1.5 for all the families with $a_1 > 1$. $\square$

4.4 Proof of Theorem 1.5 assuming $\mathbf{a_1 = 1,\ a_2 > 1}$. For this proof we apply the method of Example 4.1 to many curves. There is not enough space here to go through each of these; instead, Table 1 summarises the calculations.

Table 1: Curves excluded by surface methods

Family	Fails	Curve(s)	Method	System
7	2.3, 2.5	$\{x_0 = y_1 = y_2 = 0\}$	4.1	$\|2A - C\|$
		$\{x_0 = y_1 = z = 0\}$	4.1	$\|3A - C\|$
9	2.3	$\{x_0 = y = z_1 = 0\}$	4.1	$\|3A - C\|$
		$\{x_0 = z_1 = z_2 = 0\}$	4.1	$\|3A - C\|$
11	2.3	$\{x_0 = y_1 = z = 0\}$	4.1	$\|5A - C\|$
12	2.3, 2.5	$\{x_0 = y = z = 0\}$	4.1	$\|3A - C\|$
		$\{x_0 = y = t = 0\}$	4.1	$\|4A - C\|$
13	2.3, 2.5	$\{x_0 = y = z = 0\}$	4.1	$\|3A - C\|$
		$\{x_0 = y = t = 0\}$	4.1	$\|5A - C\|$
15	2.3	$\{x_0 = y = t = 0\}$	4.2	$\|2A - C\|$
16	2.3, 2.5	$\{x_0 = y = z = 0\}$	4.1	$\|4A - C\|$
17	2.3	$\{x_0 = y = z_1 = 0\}$	4.1	$\|4A - C\|$
20	2.5	$\{x_0 = y = z = 0\}$	4.1	$\|4A - C\|$
21	2.3	$\{x_0 = y = t = 0\}$	4.1	$\|7A - C\|$
24	2.3, 2.5	$\{x_1 = y = z = 0\}$	4.1	$\|5A - C\|$
25	2.5	$\{x_1 = y = z = 0\}$	4.1	$\|4A - C\|$
26	2.5	$\{x_0 = y = z = 0\}$	4.1	$\|5A - C\|$
29	2.3	$\{x_0 = y = t = 0\}$	4.2	$\|2A - C\|$
34	2.3	$\{x_0 = y = t = 0\}$	4.2	$\|2A - C\|$
46	2.5	$\{x_1 = y = z = 0\}$	4.1	$\|7A - C\|$

The contents of Table 1 should be interpreted as follows, in conjunction with the Big Table of [CPR]. The families listed are those with $a_1 = 1$ and $a_2 > 1$ which fail to satisfy the hypotheses of at least one of Lemmas 2.3 and 2.5; which of these two they fail is the content of the second column. Now, for a given family in the table, we run familiar arguments, in the style of the proofs of Lemmas 2.2 and 2.3, to deduce that up to coordinate change the only curves of degree at most A^3 which are not contained in $\{x_0 = x_1 = 0\}$ are those listed in the third column. (In fact, for a given family, there is usually only one of these curves up to coordinate change.)

The fourth column gives the method used to exclude the curve in question — usually that of Example 4.1, but in a few cases that of Example 4.2. Each of these methods involves picking a general surface T in some linear system with a certain base locus containing C; this linear system is given in the last column. This completes the proof. □

References

[Ch00] I. Cheltsov, Log models of birationally rigid varieties, *J. Math. Sci.*, **102**:2 (2000), pp. 3843–3875

[Ch03] I. Cheltsov, Anticanonical models of three-dimensional Fano varieties of degree four, *Mat. Sbornik*, **194**:4 (2003), pp. 147–172

[Ch05] I. Cheltsov, Elliptic structures on weighted three-dimensional Fano hypersurfaces, `arXiv:math.AG/0509324 v1` (2005)

[CP] I. Cheltsov and J. Park, Weighted Fano threefold hypersurfaces, `arXiv:math.AG/0505234 v3` (2005), to appear in *Journal für die reine und angewandte Mathematik*

[Co95] A. Corti, Factoring birational maps of 3-folds after Sarkisov, *J. Alg. Geom.*, **4** (1995), pp. 223–254

[CM] A. Corti and M. Mella, Birational geometry of terminal quartic 3-folds I, *Amer. J. Math.*, **126**:4 (2004)

[CPR] A. Corti, A. Pukhlikov and M. Reid, Fano 3-fold hypersurfaces, in *Explicit birational geometry of 3-folds*, LMS LNS **281**, A. Corti, M. Reid, editors, CUP (2000)

[Fl00] A. R. Iano-Fletcher, Working with weighted complete intersections, in *Explicit birational geometry of 3-folds*, LMS LNS **281**, A. Corti, M. Reid, editors, CUP (2000)

[FA] J. Kollár et al., *Flips and abundance for algebraic threefolds — a summer seminar at the University of Utah (Salt Lake City, 1991)*, Astérisque **211** (1992)

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varities*, Cambridge Tracts in Mathematics **134**, CUP (1998)

[Ry02] D. J. Ryder, *Elliptic and K3 fibrations birational to Fano 3-fold weighted hypersurfaces*, PhD thesis, Univ. of Warwick (2002), 122 + *vii* pages

[Ry06] D. J. Ryder, Classification of elliptic and K3 fibrations birational to some $\mathbb{Q}$-Fano 3-folds, *J. Math. Sci. Univ. Tokyo*, **13** (2006), pp. 13–42

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